P-5

VARIATIONAL METHODS FOR NONSTANDARD EIGENVALUE PROBLEMS IN MICROWAVE FIELD ANALYSIS

Ismo V. Lindell
Radio Laboratory
Helsinki University of Technology
Otakaari 5A Espoo 15,Finland 02150

Summary.

The nonstandard eigenvalue problem is defined and shown to originate in microwave field problems. A unified variational principle is introduced and applied to some simple, yet nontrivial, problems to demonstrate that a nonstandard formulation may lead to a simpler solution of the same problem than a standard one.

The eigenvalue problem.

In terms of linear operators, the eigenvalue problem can be expressed in the form

$$L(\lambda)f = 0 \tag{1}$$
$$B(\lambda)f = 0 \tag{2}$$

The problem is of the standard form for $L(\lambda)=L_0-\lambda M_0$, $B(\lambda)=B_0$, where L_0,M_0,B_0 do not depend on the parameter λ. For any other dependence, (1),(2) is termed nonstandard eigenvalue problem. Here, the operator L may be a differential or integral operator and B is an additional operator corresponding to, e.g., boundary or interface conditions of the field function f. For integral operators L the condition (2) may be nonexistant.
The nature of the parameter λ is limited in no way, it may be any physical or geometrical parameter involved in the problem. We are interested in those values of λ, for which there exist other solutions of (1),(2) than f=0, called eigenvalues. It is seen that the same problem can be conceived as an eigenvalue problem in as many different ways as there are parameters. We may wish to consider a nonstandard formulation of the problem e.g. for one of the following reasons:

- the problem cannot be formulated in a standard form
- the problem is easier to solve in a nonstandard form
- the nonstandard eigenvalue is the interesting parameter of the problem

The variational method.

A variational principle effective for eigenvalue problems can be formulated in abstract form in terms of two inner products $(\cdot,\cdot)$, $(\cdot,\cdot)_b$:

$$F(\lambda;f) \equiv (f,L(\lambda)f) + (Cf,B(\lambda)f)_b = 0 \ . \tag{3}$$

Here, C is a linear operator not dependent on λ, which can also be hidden in the definition of $(\cdot,\cdot)_b$, but generally present if the definitions of the two inner products are the same in the respective domains of the operators L and B. In all examples here, C equals 1.
It can be shown that (3) possesses roots λ that are stationary for small variations of the field function f about its correct value, i.e. solution of (1),(2), [1]. Thus, if λ can be solved from (3) in explicit form, what results is a stationary functional for λ. A necessary condition, however, is that the operator triple L,B,C is self adjoint with respect to the inner products defined, i.e., we have

$$(g,Lf) + (Cg,Bf)_b = (Lg,f) + (Bg,Cf)_b \tag{4}$$

for all f,g and the parameter λ. Non-selfadjoint problems can be written in a selfadjoint form [2].
Because the parameter λ was restricted in no way, we may identify λ with any parameter of the problem and apply (3). An explicit stationary functional is obtained if (3) is an algebraic equation for λ of the first or the second degree, but it may even be transcendental and yet possess an explicit solution.The following procedure may be helpful in applying the present method:

1. Formulate the problem, identify the operators L and B
2. Define the inner products and the operator C such that the condition (4) is satisfied
3. Apply (3) for the interesting parameter or, if it cannot be solved in explicit form, for another parameter of the problem for which (3) is solvable. In negative case, also $F(\lambda;f)$ is a stationary functional.
4. For fixed values of all parameters in the functional find the stationary value of the functional by inserting suitable test functions for f.

Examples.

The previous theory will now be elucidated with a few simple, yet nontrivial, examples. A more complete analysis of these and other examples can be found in [1].

Cutoff of a waveguide with reactance boundary.

The cutoff problem of a waveguide with a reactance boundary can be formulated in standard form writing Maxwell's equations for (1), where f represents the pair of field vectors **E**,**H** and λ is the cutoff frequency ω. This formulation is very inconvenient to apply, because we have to find reasonable test functions for a pair of field vectors. Eliminating **H**, we are left with an eigenvalue problem for the electric field **E**, which, however, is no more of a standard form in the parameter ω, because $L(\omega)$ is a quadratic function and $B(\omega)$ is a linear function. A functional can be constructed from (3) in explicit form for $\lambda=\omega$.

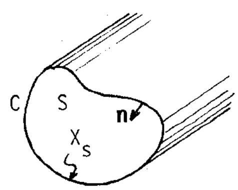

Figure 1. Waveguide with a reactance boundary.

Applying axial components of field vectors (or Hertzian potentials), a simpler formulation is obtained for the problem in terms of a scalar function E_z or H_z corresponding to the TM and TE fields of the cutoff problem. For example, for the TE field we have

$$L(k)f = (\nabla^2+k^2)H_z = 0 \qquad \text{on S} \tag{5}$$
$$B(k)f = \mathbf{n}\cdot\nabla H_z - kpH_z = 0 \qquad \text{on C.} \tag{6}$$

Here, we denote by p the normalized surface reactance X_s/η. The problem (5),(6) is clearly of the nonstandard form in the parameter k. Defining inner products as integrals over S and C, respectively, we have the operator C=1 and since (3) now is quadratic in $k=\lambda$, we can solve it for the following functionals:

$$\lambda_{\pm} = p\frac{\oint f^2 dC}{2\int f^2 dS} \pm \sqrt{\left(p\frac{\oint f^2 dC}{2\int f^2 dS}\right)^2 + \frac{\int(\nabla f)^2 dS}{\int f^2 dS}} \quad . \qquad (7)$$

The two functionals are both stationary for the correct field $f=H_z$. It is easy to see that the lower sign functional corresponds to negative cutoff frequencies and its values are obtained from the upper sign functional by reversing the sign of p and the functional. So, we may limit to one functional only.
(7) is much easier to apply than the corresponding standard functional for **E**,**H** fields, and engineering accuracy is easily obtained with a programmable calculator applying simple test functions. As a test, for a circular cylindrical waveguide with the radius a, applying the linear test function $f(\rho)=\rho+\alpha a$, where ρ is the radial distance and α a free parameter, we obtain from (7) the stationary value with respect to variations in α an accuracy better than 2.4% in the range $-\infty < p < 1$ for the TE_{01} cutoff mode.

Other choices of the eigenvalue parameter. Since we are free to choose the parameter for the eigenvalue of our problem, we might try other possibilities for λ. In the formulation of the problem (5),(6) there is another parameter, $p=X_s/\eta$, which is a nonstandard eigenvalue by the definition. In this case, the resulting functional is simpler because (3) is now a linear equation for $\lambda=p$. The solution is

$$\lambda = \frac{\int(\nabla f)^2 dS - k^2\int f^2 dS}{k\oint f^2 dC} \quad , \qquad (8)$$

and it gives us the value p as the stationary value for the correct field $f=H_z$.
The eigenvalue problem can be stated: Determine the possible boundary reactance values for which the cutoff wavenumer takes on the value k. Because of the simple functional (8), analytic approximations for the relation p(k) are possible. For the circular guide just mentioned, with the same test function, we have the result $p(k) \simeq -ka((ka)^2-18)/6((ka)^2-6)$, which is extremely tedious to deduce from the functional (7).
There exist still more parameters in this simple problem, involved in the geometry of the waveguide. For example, for the circular waveguide we have the obvious parameter $\lambda=a$. The equation (3) in this case, however, is too complicated to be solved for a in explicit form, whence we have to be satisfied either with (7) or (8). Which one of the two we choose depends on our problem. If we wish to know the functional relation between k and p, the functional (8) is more attractive because of its superior simplicity. Also, if we wish to know the p value giving us a certain k value, (8) is to be preferred. But if the problem is to find the k value for a given p, we should apply (7) because it gives us the result directly, whereas (8) has to be applied repeatedly.

Waveguide with azimuthally magnetized ferrite. A circular waveguide filled with ferrite material magnetized azimuthally to remanence with the aid of an axial current pulse has proved useful for microwave phase shifting devices [3],[4]. The operating mode is TE_{01} and the propagation factor β depends on the direction of magnetization, i.e., sign of the parameter $p=\gamma M_\phi/\omega$, where γ is the gyromagnetic ratio, and M_ϕ the magnetization in the azimuthal direction.

The pertinent equations for the axial magnetic field are in this case

$$\frac{1}{\rho}(\rho H_z')' + (k^2-k^2p^2-\beta^2-\frac{p\beta}{\rho})H_z = 0 \qquad (9)$$

$$H_z'(0) = 0 \quad , \quad H_z'(a) = 0 \qquad (10)$$

The equations (9),(10) can be solved exactly in terms of Kummer and Tricomi hypergeometric functions of pure imaginary argument and zeros for these functions have been tabulated [4]. The problem may also be attacked with variational methods. (9),(10) is clearly of the nonstandard form in all parameters k,p,β,a. The equation (3) is algebraic and quadratic in the two parameters p and β, cubic in the parameter k (note the dependence of p on k) and nonalgebraic in the geometrical parameter a. Hence, it is most advantageous to solve (3) either for p or β. The functional for $\lambda=\beta>0$ reads

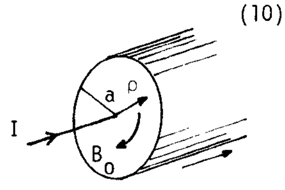

Figure 2. The ferrite-filled circular waveguide.

$$\lambda = -\frac{p\int f^2 d\rho}{2\int f^2\rho d\rho} + \sqrt{\left(\frac{p\int f^2 d\rho}{2\int f^2 \rho d\rho}\right)^2 + k^2 - p^2k^2 - \frac{\int (f')^2\rho d\rho}{\int f^2 \rho d\rho}} \qquad (11)$$

and that for p is obtained through the transformation $p \rightarrow \beta/k$.
For example, for ka=16 and $p=\pm 0.4$ we have the stationary values β/k = 0.83 and 0.92, respectively, for the simplest polynomial test function satisfying the boundary conditions (10) and containing one free parameter: $f(\rho) = \rho^3 - \frac{3}{2}\rho^2 a + \alpha$. The corresponding values from [4] are 0.86 and 0.91. The error is smaller for smaller $|p|$.

Dielectrically loaded resonator. As a final example we study a cavity of general form with a dielectric insert. The problem is relevant for microwave diagnostics, i.e., in trying to find out material parameters in terms of resonance frequency and Q measurements. The common way of formulating the problem is such that the resonance frequency, which is known, is treated as an unknown and a functional for it is constructed,[5],[6]. However, since ε_r is now the interesting parameter, we should rather treat it as an eigenvalue. A functional for $\lambda=\varepsilon_r$ can be written in the form [1]

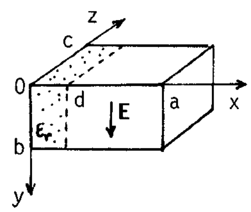

Figure 3. The resonator with an insert

$$\lambda = \frac{\int(\nabla\times\mathbf{E})^2 dV - k^2\int\mathbf{E}^2 dV + 2\oint\mathbf{n}\cdot\mathbf{E}\times(\nabla\times\mathbf{E})dS}{k^2\int\mathbf{E}^2 dV} + 1 \quad . \qquad (12)$$

Here, the integrals in the numerator are over the whole resonator volume V, whereas that in the denumerator only extends over the volume of the dielectric insert V_1.
The application of (12) is simpler than that for the resonance frequency, because for a measured value of the frequency we directly have the dielectric constant as the stationary value, whereas the other functional must be used for many times. (12) also works for lossy dielectrics if k is taken complex. Moreover, (12) is valid as well for frequency dependent media and the frequency dependence of ε is obtained if the cavity is deformed so that the resonant frequency is changed. The functional for the resonance frequency is strictly not stationary for frequency-dependent inserts. Only for slight dependence of ε on ω is it applicable.

In fact, for frequency-dependent media with known dependence, the equation (3) for $\lambda=\omega$ is more complicated to solve as that for nondispersive media.
As a test of the functional (12) we apply it for the rectangular resonator shown in Fig.3 for the basic TE_{101} mode. The result can be solved exactly from a transcendental equation. For the test function with sine dependence on the z coordinate and $x(a-x)(1+\alpha x)$ dependence on the x coordinate, we obtain the result depicted in Fig.4 for low values of ε_r.

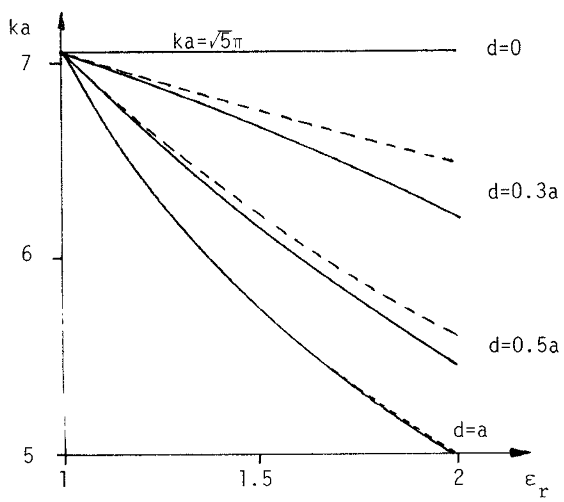

Figure 4. Relation between the dielectric constant ε_r, *the normalized resonance frequency* ka *and the thickness* d *of a dielectrically loaded rectangular resonator. Solid line: exact, dashed line: approximate.*

It is seen that this simple test function is satisfactory for low ε_r values only.
In this problem it is possible to solve (3) also for the geometrical parameter d [1].

Conclusion.

The concept of nonstandard eigenvalue problem was defined and a variational method was introduced in a very general form. Several simple examples were considered in terms of the present method and engineering accuracy was observed for very simple test functions.

References.

1 I.V.Lindell, "Variational methods for nonstandard eigenvalue problems in waveguide and resonator analysis".To be published in *IEEE Trans.Microwave Theory Tech.*

2 G.Jeng, A.Wexler, "Self-adjoint variational formulation of problems having non-self-adjoint operators".*IEEE Trans.Microwave Theory Tech.*Vol.MTT-26 no.2, pp.91-94, Feb.1978.

3 D.M.Bolle, G.S.Heller,"Theoretical considerations on the use of circularly symmetric TE modes for digital ferrite phase shifters".*IEEE Trans. Microwave Theory Tech.*Vol.MTT-13, no.7, pp.421-426, July 1966.

4 K.P.Ivanov, "Theoretical study of circular waveguide latching phasers".*Proc. 9th European Microwave Conf.*,Brighton, England,pp.373-377, 1979.

5 A.D.Berk, "Variational principles for electromagnetic resonators and waveguides".*IRE Trans. Ant. Prop.*Vol.AP-4, pp.104-111,April 1956.

6 R.F.Harrington, *Time-Harmonic Electromagnetic Fields*.New York:McGraw-Hill, 1961.